Constructions of Dirichlet structures

*Nicolas Bouleau**

Abstract. We show how the constructions of Dirichlet structures allow to equip the main probabilistic objects with Dirichlet forms. We emphasize the case of local **white** Dirichlet structures on the Poisson space and on the Wiener space. This yields tools for studying the existence of density of functionals of processes with independent increments and of stationnary processes.

1991 Mathematics Subject Classification: 31C25.

introduction

We shall use the expression 'Dirichlet structure' to denote a term $(\Omega, \mathcal{F}, m, \mathbb{D}, \mathcal{E})$ where $(\Omega, \mathcal{F}, m)$ is a measured space, most often a probability space, and $\mathcal{E}$ a Dirichlet form defined on a dense subspace $\mathbb{D}$ of $L^2(m)$, i.e. a closed positive bilinear form on which contractions operate.

Our aim is simplicity: to construct Dirichlet structures by algebraical methods which follow the classical methods of constructions of usual probability spaces. Our purpose is both pedagogical and methodological to get tools for obtaining results (existence of densities, variational calculus, etc) on the most fundamental probabilistic objects. This is an extension of Chapter V *The algebra of Dirichlet structures* of the book written with Francis Hirsch [BH].

We are studying especially **white** structures. For probabilistic structures the concept of whiteness is related to two ideas:

i) spatial independence: random variables with disjoint sets of indices are independent,

ii) stationnarity: invariance in law under translation of the index set, which is therefore a group.

For Dirichlet structures we reserve the word white for the notion which is the conjonction of the following properties:

i) spatial independence in the Dirichlet sense, that is in the sense of product of Dirichlet structures,

ii) translation invariance

iii) locality, we restrict ourselves to local Dirichlet structures, so that the functional calculus and the density criterion hold.

---

* Ecole des Ponts, Paris.

This doesn't prevent us from using sometimes whiteness in a weaker sense with only i) and ii) or even with i) weakened in spatial orthogonality.

With respect to the existing literature about Dirichlet structures on the Poisson space (or about symmetric semigroups acting on the Poisson space giving rise to a Malliavin calculus), which is numerous ([DKW], [BGJ], [W], [CP], [NV], etc) the approach is in the spirit of [BGJ] and [W] but with a presentation which allows to study more easily the criterion of existence of density. The structure on the elementary Poisson space (on $\mathbb{R}_+$) defined by Carlen and Pardoux [CP] which is local and possesses the property that the divergence operator coincides with the stochastic integral on previsible processes and satisfies the density criterion (EID) as proved in [BH] chapter V §5.3 (cf also [P]) but *it is not* white and does not extend easily to define structures for general P.I.I's.

About the case of Wiener space, the approach can be seen as an elementary introduction to the white noise theory and to the second quantization.

1. Products,Dirichlet independence

1.1. Notation and definitions

We consider first Dirichlet structures $(\Omega, \mathcal{F}, m, \mathbb{D}, \mathcal{E})$ where $(\Omega, \mathcal{F}, m)$ is a probability space and $(\mathbb{D}, \mathcal{E})$ a Dirichlet form i.e. a closed symmetric positive bilinear form with dense domain $\mathbb{D}$ in $L^2(m)$ such that

$$f \in \mathbb{D} \quad \Rightarrow \quad f \wedge 1 \in \mathbb{D} \quad \text{and} \quad \mathcal{E}(f \wedge 1) \leq \mathcal{E}(f),$$

and we suppose $1 \in \mathbb{D}$ and $\mathcal{E}(1) = 0$. To give such a structure is equivalent to consider a strongly continuous symmetric semigroup $(P_t)_{t \geq 0}$ on $L^2(m)$ which is Markovian $(0 \leq f \leq 1 \ \Rightarrow\ 0 \leq P_t f \leq 1,\ P_t 1 = 1)$ cf [BH].

Three properties concerning Dirichlet strucutres will be interesting for us:

• (OCC), (Existence of a *carré du champ* opérator)

$$\forall f \in \mathbb{D} \cap L^\infty,\ \exists \tilde{f} \in L^1,\ \forall h \in \mathbb{D} \cap L^\infty, \quad 2\mathcal{E}(fh, f) - \mathcal{E}(h, f^2) = \int \tilde{f} h \, dm.$$

Then one sets $\Gamma(f, f) = \tilde{f}$, $\Gamma(f, g)$ is defined by polarisation. Γ is a continuous operator from $\mathbb{D} \times \mathbb{D}$ into $L^1(m)$, it is uniquely defined. One has

$$\mathcal{E}(f, g) = \frac{1}{2} \int \Gamma(f, g) \, dm \quad \forall f, g \in \mathbb{D}.$$

If F is a contraction

$$\Gamma(F \circ f, F \circ g) \leq \Gamma(f, f) \quad \forall f \in \mathbb{D}.$$

- (L) (Locality)

$$\forall f \in \mathbb{D} \quad \mathcal{E}(|f+1|-1) = \mathcal{E}(f)$$

or equivalently

$$\forall f, g \in \mathbb{D} \quad fg = 0 \Rightarrow \mathcal{E}(f,g) = 0$$

Under (OCC) and (L) a functional calculus holds: If $f \in \mathbb{D}^m$, $g \in \mathbb{D}^n$, F, G Lipschitz and C^1,

$$\Gamma(F(f), G(g)) = \sum_{i,j} F'_i(f) G'_j(g) \Gamma(f_i, g_j)$$

and these hypotheses can be weakened if $m = n = 1$:

If F, G are Lipschitz from $\mathbb{R}$ to $\mathbb{R}$:

$$\Gamma(F(f), G(g)) = F'(f) G'(g) \Gamma(f, g)$$

where F' , G' are the derivatives in the sense of Lebesgue, this formula makes sense by the fact that

$$\forall f \in \mathbb{D} \quad f_*(\Gamma(f,f).m) << \lambda_1 \text{ (Lebesgue measure on } \mathbb{R}).$$

- (EID) (Energy image density property)

It is the preceding property extended to the case of dimension k:

$$\forall f \in \mathbb{D}^k \quad f_*(\det(\Gamma(f, f^*)).m) << \lambda_k$$

where $\Gamma(f, f^*)$ is the matrix $\Gamma(f_i, f_j)$.

This property is satisfied on the Wiener space when equipped with the form associated with the Ornstein-Uhlenbeck semigroup.

1.2. Products

Definition. *Let $S_i = \Omega_i, \mathcal{F}_i, m_i, \mathbb{D}_i, \mathcal{E}_i)$ $i = 1,2$ be two Dirichlet structures as defined in 1.1. The product structure $S_1 \otimes S_2$ is defined as $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, m_1 \times m_2, \mathbb{D}, \mathcal{E})$ with*

$$\mathbb{D} = \Big\{ f \in L^2(m_1 \times m_2) \;:\; \textit{for } m_2-a.e.\ y\ f(.,y) \in \mathbb{D}_1 \;\; \textit{for } m_1-a.e.\ x\ f(x,.) \in \mathbb{D}_2$$

$$\int \mathcal{E}_1(f(.,y))\, dm_2(y) \;+\; \int \mathcal{E}_2(f(x,.))\, dm_1(x) \;< +\infty \Big\}$$

and

$$\mathcal{E}(f) = \int \mathcal{E}_1(f(.,y))\, dm_2(y) \;+\; \int \mathcal{E}_2(f(x,.))\, dm_1(x).$$

It is easy to show that $S_1 \otimes S_2$ is a Dirichlet structure.
If S_1 and S_2 satisfy (OCC) then $S_1 \otimes S_2$ satisfies (OCC) too and

$$\Gamma(f)(x,y) = \Gamma_1(f(.,y))(x) + \Gamma_2(f(x,.))(y).$$

If S_1 and S_2 are local, $S_1 \otimes S_2$ is local too.

Infinite products

Definition. *Let $S_i = (\Omega_i, \mathcal{F}_i, m_i, \mathbb{D}_i, \mathcal{E}_i)$ $i \in \mathbb{N}$ be Dirichlet structures. The product $S = \bigotimes_{i\in\mathbb{N}} S_i$ is defined by $S = (\Omega, \mathcal{F}, m, \mathbb{D}, \mathcal{E})$ with*

$$(\Omega, \mathcal{F}, m) = \bigotimes_{i\in\mathbb{N}} (\Omega_i, \mathcal{F}_i, m_i) \quad \textit{(product probability space)}$$

and

$$\mathbb{D} = \Big\{ f \in L^2(m) \ : \ \forall i, \textit{ for almost all } \omega_0, \omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots$$

$$\textit{the map } x \mapsto f(\omega_0, \omega_1, \ldots, \omega_{i-1}, x, \omega_{i+1}, \ldots) \textit{ belongs to } \mathbb{D}_i$$

$$\textit{and } \int \sum_{i\in\mathbb{N}} \mathcal{E}_i(f)\, dm \ < +\infty \Big\}$$

and

$$\mathcal{E}(f) = \int \sum_{i\in\mathbb{N}} \mathcal{E}_i(f)\, dm.$$

The fact that this defines a closed form comes from the following argument:
Let (f_n) be a Cauchy sequence in $\mathbb{D}$ equipped with the norm $||\,.\,||_1$ defined by $||\,.\,||_1 = \mathcal{E} + ||\,.\,||^2_{L^2}$, such that

$$||f_n - f||_{L^2} \to 0.$$

There exists a subsequence f_{n_k} such that

$$\sum_k ||f_{n_k} - f||^2 < +\infty$$

and

$$\sum_k (\mathcal{E}(f_{n_{k+1}} - f_{n_k}))^{\frac{1}{2}} < +\infty.$$

Then for every i and for almost every $\omega_0, \omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots$ (product measure) one has

a) $\displaystyle\sum_k \int (f_{n_k}(\ldots, \omega_{i-1}, x, \omega_{i+1}, \ldots) - f(\ldots, \omega_{i-1}, x, \omega_{i+1}, \ldots))^2 \, dm_i(x) \ < +\infty$

b) $\displaystyle\sum_k (\mathcal{E}_i(f_{n_{k+1}} - f_{n_k}))^{\frac{1}{2}} \ < +\infty.$

this last fact coming from the inequalities

$$\int \mathcal{E}_i(g)^{\frac{1}{2}} \, dm \ \leq \ (\int \mathcal{E}_i(g) \, dm)^{\frac{1}{2}} \ \leq \ \mathcal{E}(g)^{\frac{1}{2}}.$$

By the closedness of the form $(\mathbb{D}_i, \mathcal{E}_i)$, it follows that for almost every $\omega_0, \omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots$ the map

$$x \mapsto f(\omega_0, \omega_1, \ldots, \omega_{i-1}, x, \omega_{i+1}, \ldots)$$

is in $\mathbb{D}_i$ and $\mathcal{E}_i(f_{n_k} - f) \to 0$, hence by the Fatou lemma

$$\int \sum_{i \in \mathbb{N}} \mathcal{E}_i(f) \ dm \ = \ \int \sum_{i \in \mathbb{N}} \lim_k \mathcal{E}_i(f_{n_k}) \ dm \ \leq \ \liminf_k \mathcal{E}(f_{n_k})$$

and since f_n is Cauchy in $\mathbb{D}$, this last term is finite and $f \in \mathbb{D}$. Similarly

$$\int \sum_{i \in \mathbb{N}} \mathcal{E}_i(f - f_n) dm = \int \sum_{i \in \mathbb{N}} \lim_k \mathcal{E}_i(f_{n_k} - f_n) dm \leq \liminf_k \mathcal{E}(f_{n_k} - f_n)$$

which can be made $\leq \varepsilon$ for k and n large enough since f_n is Cauchy in $\mathbb{D}$, so we have $f_n \to f$ in $\mathbb{D}$. After proving the closedness, the fact that the form is Dirichlet is without difficulty. □

• As for finite products if the $S_i' s$ are local so is S.
• If every S_i satisies (OCC) with carré du champ opérator Γ_i, then S satisfies (OCC) and its carré du champ operator Γ is given by

$$\forall f \in \mathbb{D} \quad \Gamma(f) = \sum_{i \in \mathbb{N}} \Gamma_i(f)$$

• About property (EID), the passage from finite to infinite dimension, which is the heart of the proof of property (EID) on the Wiener space (cf [BH]) comes from the following fact:

Proposition. *Let S_i, $i \in \mathbb{N}$ be Dirichlet structures satisfying (OCC) and (L). If each finite product*

$$S_u = \bigotimes_{i \in u} S_i \qquad u \textit{ finite part of } \mathbb{N}$$

satisfies (EID), then the infinite product

$$S = \bigotimes_{i \in \mathbb{N}} S_i$$

satisfies (EID).

Proof. . With the same notation as in the preceding proof, let $U \in \mathbb{D}^n$, then

$$\Gamma(U, U^*) = \lim_{N \uparrow \infty} \sum_{i=0}^{N} \Gamma_i(U, U^*)$$

increasing limit in the sense of semi-definite positive matrices, hence if B is a Lebesgue negligible Borel set in $\mathbb{R}^n$

$$\int 1_B \circ U . \det \Gamma(U, U^*) \, dm = \lim_N \int 1_B \circ U . \det(\sum_{i=0}^{N} \Gamma_i(U, U^*)) \, dm = 0$$

and the proposition is proved. □

1.3 Images.

We are now almost able to define the concept of D-independence. But, as in the case of probability structures, we need before to define the law (here the D-law) indeed independence means that the law of a pair is the product of the laws. Thus we need the notion of image structure:

1.3.1 Finite dimensional images.

Definition. *Let* $(\Omega, \mathcal{F}, m, \mathbb{D}, \mathcal{E})$ *be a Dirichlet structure*
(a) *for* $d \in \mathbb{N}^*$ *and* $U \in \mathbb{D}^d$, *let us define*

$$\widetilde{\mathbb{D}_U} = \{f \in L^2(U_* m) \ : \ f \circ U \in \mathbb{D}\}, \ \ \widetilde{\mathcal{E}_U}(f) = \mathcal{E}(f \circ U).$$

then $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_* m, \widetilde{\mathbb{D}_U}, \widetilde{\mathcal{E}_U})$ *is a Dirichlet dtructure and the set* Λ_d *of Lipschitz functions from* $\mathbb{R}^d$ *into* $\mathbb{R}$ *is in* $\widetilde{\mathbb{D}_U}$.
(b) *Let* $\mathbb{D}_U$ *be the closure of* Λ_d *in* $\widetilde{\mathbb{D}_U}$ *and* $\mathcal{E}_U = \widetilde{\mathcal{E}_U}\big|_{\mathbb{D}_U}$ *then the Dirichlet structure* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_* m, \mathbb{D}_U, \mathcal{E}_U)$ *is regular in the sense of Fukushima (i.e. there is a dense subspace of* $C_K(\mathbb{R}^d)$ *equipped with uniform topology, which is also dense in* $\mathbb{D}_U$*)*

The structure $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), U_* m, \mathbb{D}_U, \mathcal{E}_U)$ is called the image of S by U and denoted $U_* S$.
Remark There are cases in which $\widetilde{\mathbb{D}_U} \neq \mathbb{D}_U$.

• If S satisfies (OCC) with carré du champ operator Γ, then U_*S satisfies (OCC) with carré du champ operator Γ_U given by

$$\Gamma_U(f)(x) = \mathbb{E}[\Gamma(f \circ U)|U = x] \quad U_*m\text{-a.e.}$$

where $\mathbb{E}$ refers to the measure m. In other words

$$\mathbb{E}[g \circ U.\Gamma_U(f)] = \mathbb{E}[g \circ U.\Gamma(f \circ U)] \quad \forall \text{ bounded } g.$$

• If S is local, so is U_*S.
Under (OCC) and (L) for S we have thus

$$\forall F \in \Lambda_d \cap C^1(\mathbb{R}^d)$$

$$\Gamma_U(F)(x) = \sum_{i,j=1}^{d} \mathbb{E}[\Gamma(U_i, U_j)|U = x] \, \frac{\partial F}{\partial x_i}(x) \frac{\partial F}{\partial x_j}(x)$$

Remark. The fact that Dirichlet structures can be easily transported by images is remarkable, because the associated semigroups and the associated Markov process *do not* correspond in the same way. (cf [BH] chapter V §1.2)

1.3.2 General images.

It is not necessary the map U be in $\mathbb{D}$ so that the image by U exists. Let $S = (\Omega, \mathcal{F}, m, \mathbb{D}, \mathcal{E})$ be a Dirichlet structure. Let $(W, \mathcal{G})$ be a measurable space and U a measurable map from Ω into W.

Let us consider a set $\mathcal{A}$ of measurable maps from W into $\mathbb{R}$ such that
a) $\mathcal{A}$ is a dense subvectorspace of $L^2(U_*m)$
b) $\forall f \in \mathcal{A}$, $\forall F \in \mathcal{D}(\mathbb{R})$ $F \circ f \in \mathcal{A}$
c) $\forall f \in \mathcal{A}$, $f \circ U \in \mathbb{D}$,
then the form $(\mathcal{A}, \mathcal{E}_{\mathcal{A},U})$ defined by $\mathcal{E}_{\mathcal{A},U}(f) = \mathcal{E}(f \circ U)$ is closable in $L^2(U_*m)$. Let $(\mathbb{D}_{\mathcal{A},U}, \mathcal{E}_{\mathcal{A},U})$ its closure, the Dirichlet structure

$$U_*^{(\mathcal{A})}S = (W, \mathcal{G}, U_*m, \mathbb{D}_{\mathcal{A},U}, \mathcal{E}_{\mathcal{A},U})$$

is called the image structure of S by U with respect to $\mathcal{A}$, it depends on $\mathcal{A}$ generally.

1.4 Dirichlet independence.

Let $S = (\Omega, \mathcal{F}, m, \mathbb{D}, \mathcal{E})$ be a Dirichlet structure.

Definition 1.4.1. *$U \in \mathbb{D}^p$ and $V \in \mathbb{D}^q$ are said to be D-independent if*

$$(U, V)_*S = (U_*S) \otimes (V_*S)$$

i.e. if the image structure by the pair (U, V) is the product of the images by U and V.

Proposition 1.4.2. *U and V are D-independent iff*

$$\begin{aligned}&\forall\varphi_1,\varphi_2\in C^1(\mathbb{R}^p),\ \forall\psi_1,\psi_2\in C^1(\mathbb{R}^q)\\ &\quad\mathcal{E}(\varphi_1\circ U\psi_1\circ V,\ \varphi_2\circ U\psi_2\circ V)\\ &=\mathcal{E}(\varphi_1\circ U,\varphi_2\circ U)(\psi_1\circ V,\psi_2\circ V)_{L^2(m)}+\mathcal{E}(\psi_1\circ V,\psi_2\circ V)(\varphi_1\circ U,\varphi_2\circ U)_{L^2(m)}\end{aligned}$$

.

Under (OCC) we have the following criterion:

Proposition 1.4.3. *If S satisfies (OCC) with operator Γ, for $U\in\mathbb{D}^p$, $V\in\mathbb{D}^q$ to be independent it is necessary and sufficient that*
(a) *U and V are independent on the probability space $(\Omega,\mathcal{F},m)$*
(b) $\forall i,k\ \mathbb{E}[\Gamma(U_i,V_k)|U,V]=0\ m-a.e.$
(c) $\forall i,j\ \mathbb{E}[\Gamma(U_i,U_j)|U,V]=\mathbb{E}[\Gamma(U_i,U_j)|U]\ m-a.e.$
(d) $\forall k,l\ \mathbb{E}[\Gamma(V_k,V_l)|U,V]=\mathbb{E}[\Gamma(V_k,V_l|V]\ m-a.e..$

Remark 1.4.4. *A sufficient condition for U and V to be independent is that*
(a) *U and V are independent*
(b') $\Gamma(U_i,V_k)=0\ \forall i,k$
(c') $(U,\Gamma(U_i,U_j))$ *is independent of V*
(d') $(V,\Gamma(V_i,V_j))$ *is independent of U.*

D-independence can easily be extended to infinite families.

2. White structures related to the Poisson space

We shall first construct local Dirichlet structures on a Poisson space over a measurable space $(X,\mathcal{X})$ without other properties for the moment. It will yield white structures in the sense of i) and iii) only. After that we shall look at what happens when $X=Y\times\mathbb{R}_+$, getting structures associated with general processes with independent increments.

2.1. Finite Poisson space

Let us first recall the construction of the finite Poisson space, that is in other words of the Poisson point process with finite intensity measure.

Let $(X,\mathcal{X})$ be a measurable space and μ a finite positive measure on $(X,\mathcal{X})$. Let us set $\theta=\mu(X)$, $\mu_0=\frac{1}{\theta}\mu$. Let us consider the product probability space

$$(X,\mathcal{X},\mu_0)^{\mathbb{N}^*}\times(\mathbb{N},\mathcal{P}(\mathbb{N}),\mathbb{P}_\theta)$$

where $\mathbb{P}_\theta$ is the Poisson law on $\mathbb{N}$ with parameter θ:

$$\mathbb{P}_\theta\{n\} = e^{-\theta}\frac{\theta^n}{n!} \quad n \in \mathbb{N}.$$

Let $(X_n)_{n\in\mathbb{N}^*}$, Y be the coordinate mappings from this product space into its factors. So X_n are independent variables with values in X and law μ_0 and Y is independent of the X_n's with law $\mathbb{P}_\theta$. The formula

$$N = \sum_{n=1}^{Y} \delta_{X_n} \qquad (\text{with } \sum_1^0 = 0)$$

defines a random variable N with values in the set M_p of point measures on $(X,\,\mathcal{X})$, that is sums of Dirac masses, equipped with the σ-field $\mathcal{M}_p$ generated by the maps

$$m \longrightarrow m(A) \qquad A \in \mathcal{X},$$

and N has the following properties (cf Neveu [N])

a) $\forall A \in \mathcal{X}$, the random variable $N(A)$ has a Poisson law with parameter $\mu(A)$

$$\mathbb{P}(N(A) = k) = e^{\mu(A)}\frac{\mu(A)^k}{k!}$$

In particular $\mathbb{E}N(A) = \mu(A)$. (so μ is the intensity measure of N).

b) For $B_1, \ldots, B_n \in \mathcal{X}$ pairwise disjoint, the random variables $N(B_1), \ldots, N(B_n)$ are independent.

We follow this construction starting from a Dirichlet structure $(X, \mathcal{X}, \mu_0, \mathbf{d}, \mathbf{e})$, which, as before, is supposed to satisfy $1 \in \mathbf{d}$, $\mathbf{e}(1) = 0$, and in addition sat isfies (L) and (OCC) with operator γ.

The product Dirichlet structure (2.1.1):

$$(\Omega,\, \mathcal{F},\, m,\, \mathbb{D},\, \mathcal{E}) = (X, \mathcal{X}, \mu_0, \mathbf{d}, \mathbf{e})^{\mathbb{N}^*} \times (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathbb{P}_\theta, L^2(\mathbb{P}_\theta), 0)$$

as we have seen satisfies (L) and (OCC). Let still $(X_n)_{n\in\mathbb{N}^*}$ and Y be the coordinate mappings and

$$N = \sum_{n=1}^{Y} \delta_{X_n}$$

the associated Poisson point process.

Lemme 2.1.2. *Let $U = F(Y, X_1, X_2, \ldots, X_n, \ldots)$ be in $L^2(\Omega, \mathcal{F}, m)$,*

(a) $U \in \mathbb{D}$ *iff*

$$\forall m \in \mathbb{N}\ F(m, X_1, X_2, \ldots) \in \mathbb{D} \text{ and } \mathcal{E}(U) = \int \mathcal{E}(F(m, \ldots))\, d\mathbb{P}_\theta(m) < +\infty$$

(b) $U \in \mathbb{D}$ *iff*

$$\forall m \forall k,\ \text{for } \mu_0^{\otimes\mathbb{N}^*} - a.e.\ x_1, \ldots, x_{k-1}, x_{k+1}, \ldots$$

$$F(m, x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots) \in \mathbf{d}$$

and

$$\mathcal{E}(U) = \frac{1}{2}\int \sum_{k=1}^{\infty} \gamma_k(F(m, x_1, \dots, x_{k-1}, ., x_{k+1}, \dots))\, d\mathbb{P}_\theta(m)\, d\mu_0^{\otimes \mathbb{N}^*} < +\infty$$

(c) *For $U \in \mathbb{D}$ the carré du champ operator Γ of $(\Omega, \mathcal{F}, m, \mathbb{D}, \mathcal{E})$ is given by*

$$\Gamma(F) = \sum_{k=1}^{\infty} \gamma_k(F(Y, X_1, \dots, X_{k-1}, ., X_{k+1} \dots))(X_k)$$

This is a direct consequence of properties of product structures.

Lemma 2.1.3. *For $f \in \mathbf{d}$, $N(f)$ is defined and in $\mathbb{D}$.*

Proof. f is an equivalence class for μ_0 or μ-a.e. equality, now, if $f = g$ μ-a.e.

$$\mathbb{E}|N(f) - N(g)| \le \mathbb{E}N|f-g| = \mu|f-g| = 0,$$

so $N(f) = N(g)$ μ-a.e.

Using the Laplace functional of N, (cf [N])

$$\mathbb{E}e^{-\lambda N(f)} = \exp\{-\int (1-e^{-\lambda f})\, d\mu\} \qquad \lambda \ge 0$$

we get

$$\mathbb{E}N(f)^2 = \int f^2\, d\mu + (\int f\, d\mu)^2$$

proving that $N(f) \in L^2(m)$ for $f \in L^2(\mu)$.

Then for $f \in \mathbf{d}$,

$$\begin{aligned}\Gamma(N(f)) &= \sum_{k=1}^{\infty} \gamma_k(\sum_{n=1}^{Y} f(X_n)) \\ &= \sum_{k=1}^{\infty} 1_{\{k \le Y\}} \gamma(f)(X_k) = \sum_{k=1}^{Y} \gamma(f)(X_k) \\ &= N(\gamma(f))\end{aligned}$$

and $\mathcal{E}(N(f)) = \frac{1}{2}\mu(\gamma(f)) = e(f)$ □

So we have obtained (2.1.4):

$$\begin{array}{c} \text{for } f \in \mathbf{d} \quad \Gamma(N(f)) = N(\gamma(f)) \\ \text{for } f, g \in \mathbf{d} \quad \Gamma(N(f), N(g)) = N(\gamma(f,g)) \end{array}.$$

Corollary 2.1.5. *If $fg = 0$ and if $f, g \in \mathbf{d}$, then $N(f)$ and $N(g)$ are D-independent.*

Proof. We know that $N(f)$ and $N(g)$ are independent. By remark 1.4.4 it suffices to show

a) $\gamma(N(f),N(g))=0$, which comes from (2.1.4) because $\gamma(f,g)=0$

b) $(N(f),\Gamma(N(f)))$ is independent of $N(g)$, which comes from (2.1.4) too, since $(N(f),\Gamma(N(f)))=(N(f),N(\gamma(f)))$ and $\gamma(f)=0$ on $\{g\neq 0\}$ by the fact that the structure $(X,\mathcal{X},\mu_0,\mathbf{d},\mathbf{e})$ is local. □

Remark 2.1.6. If $\forall m \quad (X,\mathcal{X},\mu_0,\mathbf{d},\mathbf{e})^m$ satisfies (EID) then $(\Omega,\ \mathcal{F},\ m,\ \mathbb{D},\ \mathcal{E})$ satisfies (EID).

Remark 2.1.7. The map

$$\widetilde{N}\,:\; f\longrightarrow (N(f)-\int f\,d\mu)$$

from $L^2(X,\mathcal{X},\mu_0)$ into $L^2(\Omega,\mathcal{F},m)$ is an homomorphism which preserves L^2-norms and Dirichlet forms, and is therefore an isometry from $\mathbf{d}$ into $\mathbb{D}$ equipped by any α-norm: $(\alpha\|\,.\,\|^2_{L^2(\mu_0)}+\mathbf{e})^{\frac{1}{2}}$ and $(\alpha\|\,.\,\|^2_{L^2(m)}+\mathcal{E})^{\frac{1}{2}}$. Indeed we have

$$\mathbb{E}(N(f)-\int fd\mu)^2=\int f^2d\mu$$

$$\mathcal{E}(N(f)-\int fd\mu)=\mathbf{e}(f) \text{ for } f\in\mathbf{d}.$$

From f and g with disjoint support $\widetilde{N}$ gives $\widetilde{N}(f)$ and $\widetilde{N}(g)$ which are D-independent. $\widetilde{N}$ is therefore a Wiener-Karhunen measure in the strong Dirichlet sense.

Remark 2.1.8 Let us recall that M_p is the set of point measures on $X,\mathcal{X}$ with the σ-field $\mathcal{M}_p$ generated by the maps $m\to m(A) \quad A\in\mathcal{X}$. We can consider the image of the structure $(\Omega,\ \mathcal{F},\ m,\ \mathbb{D},\ \mathcal{E})$ by N in the sense of general images of §1.2.3 with respect to the set $\mathcal{A}$ of measurable functions from $(M_p,\mathcal{M}_p)$ into $(\mathbb{R},\mathcal{B}(\mathbb{R}))$:

$$\mathcal{A}=\{m\to G(m(f_1),\ldots,m(f_d)),\ G\in\mathcal{D}(\mathbb{R}^d),\ f_1,\ldots,f_d\in\mathbf{d},d\in\mathbb{N}^*\}$$

the required properties on $\mathcal{A}$ are fulfilled, as easily seen. The image structure is

$$(\mathcal{M}_p,\ \mathcal{M}_p,\ \mathbb{P},\ \mathbb{D}_1,\ \mathcal{E}_1)$$

where $\mathbb{P}$ is the law of a Poisson point process with intensity μ, $(\mathbb{D}_1,\ \mathcal{E}_1)$ is a local Dirichlet form satisfying (OCC) and characterized by the fact that its carré du champ operator Γ_1 verifies:

$$\Gamma_1(F(N(f_1),\ldots,N(f_d)))=\sum_{i,j}F'_i(N(f_1),\ldots,N(f_d))F'_j(N(f_1),\ldots,N(f_d))N(\gamma(f_i,f_j))$$

$\forall f_i\in\mathbf{d},\ i=1,\ldots,,\forall F\in\mathcal{D}(\mathbb{R}^d)$.

2.2. σ-finite Poisson space.

The probabilistic definition of a Poisson point process with σ-finite intensity measure μ is easily made by product using property b) of the Poisson point process.

In the case of Dirichlet structures, the construction cannot be so simple.

Remark 2.2.1. We have defined for the moment Dirichlet structures only on probability spaces. It is possible (cf [BH]) with minor changes to define structures $(E, \mathcal{F}, m, \mathbb{D}, \mathcal{E})$ with m σ-finite. Hypotheses (L) and (OCC) are expressed in the same way.

2.2.2. We shall assume that Dirichlet structures $(X, \mathcal{X}, \mu_k, \mathbf{d}_k, \mathbf{e}_k)$ are given where μ_k are σ-finite measures such that $\mu = \sum_k \mu_k$ is σ-finite and that $\cap_k \mathbf{d}_k$ is dense in $L^2(\mu)$. If we define the form $e = \sum_k \mathbf{e}_k$ on $\cap_k \mathbf{d}_k$, it is easy to see that it is closable and therefore defines a Dirichlet structure

$$(X, \mathcal{X}, \mu, \mathbf{d}, \mathbf{e}).$$

We suppose also that the structures $(X, \mathcal{X}, \mu_k, \mathbf{d}_k, \mathbf{e}_k)$ are local, satisfy (OCC) with the same operator in the sense that their carré du champ operator coincides with that of $(X, \mathcal{X}, \mu, \mathbf{d}, \mathbf{e})$ on $\mathbf{d}$. It is denoted by γ.

By the construction of remark 2.1.8 with each of these structures it is possible to define a Poisson point process N_k with intensity measure μ_k defined on a Dirichlet structure S_k given by formula (2.1.1) with carré du champ operator Γ_k satisfying:

$$\begin{aligned}
&\text{For } U = F(N_k(f_1), \ldots, N_k(f_d)),\ F \in \mathcal{D}(\mathbb{R}^d),\ f_i \in \mathbf{d} \\
&\Gamma_k(U) = \sum_{i,j} F_i'(N_k(f_1), \ldots, N_k(f_d)) F_j'(N_k(f_1), \ldots, N_k(f_d)) N_k(\gamma(f_i, f_j))
\end{aligned}$$

Let us consider the product $S = \bigotimes_k S_k$. Let us set

$$S = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{D}}, \widetilde{\mathcal{E}}).$$

As it is often done in probability theory, to avoid the use of the coordinate mappings in the notation, we shall consider that the N_k's are random variables defined on $\widetilde{\Omega}$ depending only on the coordinate map of order k. So the N_k's are independent and D-independent random variables defined on $S = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{D}}, \widetilde{\mathcal{E}})$.

Then we define $\widetilde{N}$ by

$$\widetilde{N} = \sum_k N_k$$

we know already by well known probabilistic arguments, that $\widetilde{N}$ is a Poisson point process with intensity measure $\mu = \sum_k \mu_k$.

The structure S possesses a carré du champ operator $\widetilde{\Gamma}$ which is the sum of the Γ_k's acting on the k-th coordinate mapping.

If $U = F(\widetilde{N}(f_1), \dots, \widetilde{N}(f_d))$ $f_i \in \mathbf{d}$, $i = 1, \dots, d$, $F \in \mathcal{D}(\mathbb{R}^d)$, one has (2.2.2.1):

$$\begin{aligned}\widetilde{\Gamma}(U) &= \sum_k \sum_{i,j} F_i'(\widetilde{N}(f_1), \dots, \widetilde{N}(f_d)) F_j'(\widetilde{N}(f_1), \dots, \widetilde{N}(f_d)) N_k(\gamma(f_i, f_j)) \\ &= \sum_{i,j} F_i'(\widetilde{N}(f_1), \dots, \widetilde{N}(f_d)) F_j'(\widetilde{N}(f_1), \dots, \widetilde{N}(f_d)) \widetilde{N}(\gamma(f_i, f_j))\end{aligned}$$

Finally, the image of the structure S by $\widetilde{N}$ with respect to the set $\mathcal{A}$ of measurable mappings from $(M_p, \mathcal{M}_p)$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$$\mathcal{A} = \{m \to G(m(f_1), \dots, m(f_d)),\ G \in \mathcal{D}(\mathbb{R}^d),\ f_1, \dots, , f_d \in \mathbf{d},\ d \in \mathbb{N}^*\}$$

introduced in 2.1.8, is a Dirichlet structure

$$(\mathcal{M}_p,\ \mathcal{M}_p,\ \mathbb{P},\ \mathbb{D}_1,\ \mathcal{E}_1)$$

which is local, satisfies (OCC) with operator Γ such that, if we note N the indentity mapping from M_p into M_p,

. N is a Poisson point process with intensity μ

. for $U = F(N(f_1), \dots, N(f_d))$ $f_i \in \mathbf{d}$, $i = 1, \dots, d$, $F \in \mathcal{D}(\mathbb{R}^d)$,

$$\Gamma(U) = \sum_{i,j} F_i'(N(f_1), \dots, N(f_d)) F_j'(N(f_1), \dots, N(f_d)) N(\gamma(f_i, f_j)).$$

In particular if $f \in \mathbf{d}$

$$\Gamma(N(f)) = N(\gamma(f))$$

and $$\mathcal{E}(N(f)) = \frac{1}{2}\mathbb{E}\Gamma(N(f)) = \frac{1}{2}\int \gamma(f)\, d\mu = \mathbf{e}(f)$$

If $f \in \mathbf{d} \cap L^1(\mu)$ one has

$$\mathbb{E}(N(f) - \int f d\mu)^2 = \int f^2 d\mu$$

$$\mathcal{E}(N(f) - \int f d\mu) = \mathbf{e}(f).$$

By these relations, the compensated Poisson process $N - \mu$ is defined on $L^2(\mu)$ and is an isometry in the Dirichlet sense as before.

2.3 Dirichlet structures related to a process with independent increments (PII)

We apply the preceding study to the case where

$$\begin{aligned}&X = \mathbb{R}_+ \times Y \\ &\mathcal{X} = \mathcal{B}(\mathbb{R}_+) \times \mathcal{Y} \qquad \mathcal{Y}\ \sigma-\text{field on } Y\end{aligned}$$

and where

$$\mu = dt \times n$$

with n σ-finite measure on $(Y, \mathcal{Y})$.

We are given a Dirichlet structure on $(X, \mathcal{X})$ say $(X, \mathcal{X}, \mu, \mathbf{d}, \mathbf{e})$ which is local (L) and possesses a carré du champ operator (OCC) denoted γ. By the preceding construction one obtains a structure $(\mathcal{M}_p,\, \mathcal{M}_p,\, \mathbb{P}_\mu,\, \mathbb{D},\, \mathcal{E})$ and the indentity mapping N is a Poisson point process with intensity μ.

This contains the case of processes with independent increments(Y_t), if we put

$$N = \sum_{s \in \mathbb{R}_+} \delta_{(s, Y_s - Y_{s-})}$$

this is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^*$ with intensity $dt \times \nu$ where ν is the Lévy measure of the PII (Y_t).

It is now possible to develop the stochastic calculus related to this PII (Y_t) (stochastic integrals, s.d.e., etc) to study the belonging of the obtained random variables to $\mathbb{D}$, and applying remark 2.1.6 and (EID) to obtain density results.

3. White structures related to the Wiener space

3.1

Let us recall the construction of the Wiener integral and that of Brownian motion it gives. Let (χ_n) be an orthonormal basis of $L^2(T, \mathcal{T}, \mu)$, where $(T, \mathcal{T}, \mu)$ is a σ-finite measured space, (g_n) be a sequence of independent reduced Gaussian random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. With $f \in L^2(T, \mathcal{T}, \mu)$ we associate $I(f) \in L^2(\Omega, \mathcal{A}, \mathbb{P})$ by

$$I(f) = \sum_n < f, \chi_n > g_n .$$

I is a homomorphism from $L^2(T, \mathcal{T}, \mu)$ into $L^2(\Omega, \mathcal{A}, \mathbb{P})$. If f and g are in L^2 and if $\int f g d\mu = 0$ then $I(f)$ and $I(g)$ are independent.

Let us take $T = [0, 1]$, $\mathcal{T} = \mathcal{B}([0, 1])$, $\mu = dt$, and let us put

$$B(t) = \sum_n \psi_n(t) g_n \quad \text{with} \quad \psi_n(t) = \int_0^1 \chi_n(s)\, ds$$

this series converges in $\mathcal{C}([0, 1])$ a.s. and in $L^p((\Omega, \mathcal{A}, \mathbb{P}), \mathcal{C}([0, 1]))$ for $p \in [1, \infty[$ (cf [BH] chapter V example 1.3.2) and defines a Brownian motion. We put

$$I(f) = \int_0^1 f(s)\, dB_s .$$

3.2

The preceding construction involves the product probability space

$$(\Omega, \mathcal{A}, \mathbb{P}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), N(0,1))^{\mathbb{N}}$$

the $g_n's$ being the coordinate mappings. If we equip each factor of this product with a Dirichlet form $(\mathbb{R}, \mathcal{B}(\mathbb{R}), N(0,1), \mathbf{d}_n, \mathbf{e}_n)$ which we suppose, as usual, local and with (OCC) γ_n, by the theorem on products we obtain a Dirichlet structure on $(\Omega, \mathcal{A}, \mathbb{P})$

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \mathcal{E}) = \prod_n (\mathbb{R}, \mathcal{B}(\mathbb{R}), N(0,1), \mathbf{d}_n, \mathbf{e}_n)$$

which is local and with (OCC) Γ such that

$$\forall U \in \mathbb{D} \quad U = F(g_0, g_1, \ldots, g_n, \ldots)$$

$$\Gamma(U) = \sum_n \gamma_n(F)$$

where the γ_n acts on the n-th argument.

Examples.

3.2.1. Let us take

$$\begin{aligned} \gamma_n(u) &= u'^2 \\ \mathbf{e}_n(u) &= \frac{1}{2}\int u'^2 \, dN(0,1) \\ \mathbf{d}_n &= H^1(\mathbb{R}, N(0,1)). \end{aligned}$$

we obtain

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), N(0,1), H^1(\mathbb{R}, N(0,1)), u \to \frac{1}{2}\int u'^2 \, dN(0,1))^{\mathbb{N}}$$

and this structure is an abstract Wiener space equipped with the Ornstein-Uhlenbeck structure since we have

$$\begin{aligned} \Gamma\left(\int_0^1 f(s)dB_s\right) &= \Gamma\left(\sum_n < f, \chi_n > g_n\right) = \sum < f, \chi_n > \gamma(g_n) \\ &= \sum < f, \chi_n >= ||f||^2_{L^2[0,1]} = \int_0^1 f^2(s)ds \end{aligned}$$

3.2.2. Let us take

$$\chi_n(t) = e^{2i\pi nt}, \quad n \in \mathbf{Z}$$

$$\gamma_n(u) = a(n)u'^2$$

$$\mathbf{e}_n(u) = \frac{1}{2}a(n)\int u'^2 \, dN(0,1)$$

$$\mathbf{d}_n = H^1(\mathbb{R}, N(0,1))$$

we get

$$(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \mathcal{E}) = \prod_n (\mathbb{R}, \mathcal{B}(\mathbb{R}), N(0,1), H^1(\mathbb{R}, N(0,1)), u \rightarrow \frac{1}{2}a(n)\int u'^2 \, dN(0,1))$$

a) For $a(n) = 4\pi n^2$ we have for $f \in L^2[0,1]$

$$f(t) = \sum_{n \in \mathbf{Z}} \hat{f}_n e^{2i\pi nt}$$

$$\int_0^1 f(s)dB_s = \sum \hat{f}_n g_n$$

$$\Gamma(\int_0^1 f(s)dB_s) = \sum_n \hat{f_n}^2 4\pi n^2 = \|f'\|^2_{L^2[0,1]}$$

and by the theorem on products we see that $\int_0^1 f(s)dB_s$ belongs to $\mathbb{D}$ if and only if

$$f \in L^2[0,1] \quad \text{and} \quad \sum_n \hat{f_n}^2 4\pi n^2 < +\infty$$

that is f is continuous on the torus T^1 and belongs to the Sobolev space $H^1(T^1)$.
b) For $a(n) = 2\pi n^{2q}$ $q \in \mathbb{N}$, we obtain similarky a local Dirichlet structure with carré du champ operator Γ satisfying

$$\Gamma(\int_0^1 f(s)dB_s) = \int_0^1 {f^{(q)}}^2(s)ds$$

and $\int_0^1 f(s)dB_s \in \mathbb{D}$ iff $\sum_{n\in\mathbf{Z}} n^{2q}\hat{f_n}^2 < +\infty$.

Whiteness.

The structures obtained in 3.1.2 are **white** because if f, g are such that $\sum_{n\in\mathbf{Z}} n^{2q}(\hat{f_n}^2 + \hat{g_n}^2) < +\infty$ and if

$$fg = 0$$

then $U = \int f(s)dB_s$ and $V = \int g(s)dB_s$ are D-independent. Indeed, by remark 1.4.4, it suffices to show

$$\begin{cases} \Gamma = 0 \\ (U, \Gamma(U)) \quad \text{is independent of } V \\ (V, \Gamma(V)) \quad \text{is independent of } U \end{cases}$$

But $\Gamma(U,V) = \int f^{(q)}(s)g^{(q)}(s)\,ds = 0$ by the fact that if $u, v \in H^1(T^1)$ $uv = 0 \Rightarrow u'v' = 0$ a.e. and the two other properties are fulfilled because $\Gamma(U)$ and $\Gamma(V)$ are constants.

(EID).

Since the energy image density property is fulfilled on finite products, it follows that all structures obtained in 3.1.2 satisfy (EID).

Gradient and semigroup.

Let $(\Omega, \mathcal{A}, \mathbb{P}, \widetilde{\mathbb{D}}, \widetilde{\mathcal{E}})$ be the Dirichlet structure obtained in 3.2.1.a). By looking on the multiple Wiener integrals it is easy to show that this structure possesses a gradient $\widetilde{D}$ (cf [BH] chapter V §5.2) with the Hilbert space $H = L^2[0,1]$.

Denoting by D, δ, A, the gradient, the divergence, the generator, of the classical Ornstein-Uhlenbeck structure (cf Yan [Y]) we have:

$$\widetilde{D} = \frac{d}{dt}D$$
$$\widetilde{\delta} = -\delta\frac{d}{dt}$$
$$\widetilde{A} = \frac{1}{2}\delta\frac{d^2}{dt^2}D$$

Let $p_t$be the heat semigroup on T^1, then the semigroup $\widetilde{P}_t$ associated with the structures

$$(\Omega, \mathcal{A}, \mathbb{P}, \widetilde{\mathbb{D}}, \widetilde{\mathcal{E}})$$

is characterized by its action on multiple Wiener integrals:

$$\widetilde{P}_t(I_k(f)) = I_k(p_t^{\otimes k} f)$$

where $f \in H^1_{sym}(T^k)$.

Looking then to the exponential vectors

$$Exp(h) = \exp(\int h(s)dB_s - \frac{1}{2} < h, h >)$$

we obtain

$$\widetilde{P}_t Exp(h) = Exp(p_t h)$$

that is $(\widetilde{P}_t)$ is obtained by the second quantization and we have the Mehler formula (cf [FLP])

$$\widetilde{P}_t F(\omega) = \mathbb{E}_w[F(p_t\omega + (I - p_{2t})^{\frac{1}{2}} w)].$$

and the similar results hold for the other structures of example 3.2.2.

References

[BGJ] Bichteler K., Gravereaux J. B., Jacod J., *malliavin calculus for processes with jumps* Gordon and Breach (1987)

[BH] Bouleau, N., Hirsch, F., *Dirichlet forms and Analysis on Wiener space* de Gruyter (1991)

[CP] Carlen E. A., Pardoux E., Differential calculus and integration by parts on Poisson space, in *Stochastics, algebra and analysis in classical and quantum mechanics*,p63-73, Kluwer (1990)

[DKW] [redacted]rmoune A., Kree P., Wu L., Calcul stochastique non adapté par rapport à la mesure aléatoire de Poisson, Lect. notes in M. 1321, P 477-484, Springer (1980)

[FLP] Feyel, D., La Pradelle, A. de,Opérateurs linéaires gaussiens, Potential Analysis 3 (1994), 89-105.

[N] Neveu, J., Processus ponctuels, Ecole d'été de Saint Flour, lect. notes in M. 598, Springer (1977)

[NV] Nualard, D., Vives, J., Anticipative calculus for the Poisson process based on the Fock space, sem Prob. XXIV, lect. notes in M. 1426, p154-165, Springer (1990)

[P] Privault N., Calcul chaotique et variationnel pour le processus de Poisson, thesis, Univ Paris 6 (1994)

[W] Wu L., Construction de l'operateur de Malliavin sur l'espace de Poisson, Sem. Prob. XXI, p100-113, Springer (1987)

[Y] Yan Y. a., Development des distributions suivant les chaos de Wiener et application à l'analyse stochastique, sem Prob. XXI, p27 et seq, Springer (1987)